Journal of Sound and Vibration (1997) **205**(2), 135–150

TORSIONAL VIBRATION OF CRANKSHAFTS: EFFECTS OF NON-CONSTANT MOMENTS OF INERTIA

E. BRUSA, C. DELPRETE AND G. GENTA

Dipartimento di Meccanica, Politecnico di Torino, I-10129 Torino, Italy

(*Received* 10 *April* 1995, *and in final form* 17 *February* 1997)

The dynamic analysis of the running hardware of reciprocating machines is complex and is usually dealt with by using a number of simplifying assumptions. Usually an "equivalent dynamic system" is built for performing the torsional analysis: such equivalent system has inertial properties which are assumed to be constant and the variation of the actual configuration is taken into account only by adding suitable "inertia torques" to the driving torques. The aim of the present paper is that of studying the torsional vibration of crankshafts with account taken of the variation of the geometry of the system with the crank angle; both the free behaviour and the response to external excitation are dealt with. The analysis is linearized and a mathematical model having the form of a set of linear differential equations with periodic coefficients is obtained. The solution of the free behaviour is obtained through a formulation similar to Hill's infinite determinant, whose truncated forms yield an approximated solution of accuracy increasing with the number of harmonics which are retained. A similar method allows the forced response to be computed. Two examples, one related to a simplified single-cylinder machine and one to an actual aircraft engine, conclude the work.

1. INTRODUCTION

The computation of dynamic stressing crankshafts is not straightforward. The geometry of such elements couples the torsional, flexural and axial behaviour and, owing to the presence of crank mechanisms, the configuration of the system changes in time, with a period equal to one revolution. Also the forces acting on the system have a history which, although being periodic, is usually quite complicated. Generally speaking, the forces can be expressed in the form of a Fourier series, but the number of harmonics which must be retained is quite high and consequently the possibilities of resonance are many.

The dynamic behaviour of crankshafts has been studied in great detail since the thirties and very good and detailed handbooks on the subject are available [1, 2]. As a result of such studies dynamic failures of crankshafts, which were plaguing internal combustion engines, became a rare accident. Nevertheless the mathematical models usually employed are based on a number of assumptions and an extensive experimentation is still needed to compensate for the inadequacy of the dynamic analysis.

In many cases the most dangerous vibrations are those linked with modes which are essentially torsional and consequently the present paper deals only with torsional vibration of crankshafts. It must be however remembered that the very assumption of uncoupling can be a rough approximation and that all vibration of reciprocating machines involves all degrees of freedom of the system.

0022–460X/97/320135 + 16 $25.00/0/sv970964

The aim of the present paper is that of developing a computational procedure that can overcome some of these inadequacies by using a model which is at any rate linearized and therefore cannot take into account clearances and the presence of oil at the crank and wrist pins. Moreover the present model is at any rate based on the uncoupling of the torsional behaviour from axial and lateral dynamics and, when any actual system is studied, suffers from the lack of a suitable model for damping. The last difficulty can be particularly important in both the assessment of the stability of the system and the dynamic stressing of the shaft.

2. ANALYSIS

2.1. TRADITIONAL APPROACH FOR TORSIONAL VIBRATION OF RECIPROCATING MACHINES

The crankshaft and the reciprocating elements are usually reduced to a lumped parameters ‘‘equivalent system’’ consisting of a straight massless shaft on which a number of moments of inertia are fitted. Only the degrees of freedom related to torsional rotations of the shaft are involved. The equivalent system of the reciprocating machine is then coupled with the model of the devices connected to the crankshaft, which are as well modelled as lumped parameter systems. The equivalent model is based on a straight shaft and its torsional behaviour is consequently uncoupled from the axial and the flexural ones.

To build the equivalent model, the inertia of the reciprocating elements and that of the cranks are lumped in a number of moments of inertia which are connected to each other by a number of straight shafts, with circular or annular cross-section, having the actual diameter of the shaft or a conventional reference diameter and an ‘‘equivalent’’ length simulating the torsional stiffness of the relevant part of the shaft. To build the equivalent model the moments of inertia of the flywheels simulating the crank-connecting rod-piston systems and the equivalent lengths of the shaft must be computed.

Following the scheme reported in reference [3], one finds that the total kinetic energy of the crank and the reciprocating elements (see Figure 1) is (a list of symbols is given in the Appendix)

$$\mathscr{T} = \tfrac{1}{2} J_{eq}\dot{\theta}^2. \tag{1}$$

The whole system can thus be replaced by a single moment of inertia J_{eq}, variable with the crank angle θ, equal to

$$J_{eq} = J_d + m_1 r^2 + (m_2 + m_p) r^2 f_1(\theta) + J_0 f_2(\theta), \tag{2}$$

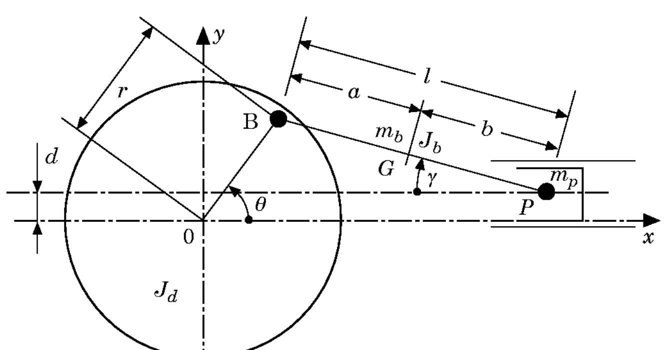

Figure 1. Sketch of the crank, connecting rod, piston mechanism. $\alpha = r/l$, $\beta = d/l$.

where J_d and m_p are the moment of inertia of the crank and the mass of the piston and m_1, m_2 and J_0 are two masses and a moment of inertia used to simulate the inertial properties of the connecting rod [3].

Functions $f_1(\theta)$ and $f_2(\theta)$ depend only on two nondimensional parameters: the ratio $\alpha = r/l$, which is always smaller than 1 and practically never exceeds 0·3, and ratio $\beta = d/l$, which has been introduced to take into account the possibility that the axis of the cylinder is skew with the axis of the crankshaft. Also β is usually very small (not greater than 0·045) and in many cases vanishes. The expressions for the two functions are

$$f_1(\theta) = [\sin(\theta) + \alpha(\sin(2\theta)/2\cos(\gamma)) - \beta(\cos(\theta)/\cos(\gamma))]^2, \tag{3}$$

$$f_2(\theta) = \alpha^2[\cos(\theta)/\cos(\gamma)]^2. \tag{4}$$

Angles γ and θ are linked by the relationship

$$r\sin(\theta) = d + l\sin(\gamma): \tag{5}$$

i.e.,

$$1/\cos(\gamma) = \frac{1}{\sqrt{1 - [\alpha\sin(\theta) - \beta]^2}}. \tag{6}$$

Functions $f_1(\theta)$ and $f_2(\theta)$ can easily be expressed as Fourier series whose coefficients are power series of parameters α and β. In such series usually six or seven harmonics in θ and powers of α and β up to fifth or sixth are considered. The expressions of the coefficients so obtained become quickly quite intricate when the number of coefficients which are considered is increased. At present it is not necessary to express the series for functions $f_1(\theta)$ and $f_2(\theta)$: it is straightforward to compute numerically the equivalent moment of inertia for a number of values of θ by using equations (2–4) and then to compute a numerical Fourier transform in the form

$$J_{eq} = J_{c_0} + \sum_{j=1}^{r} J_{c_j}\cos(j\theta) + \sum_{j=1}^{r} J_{s_j}\sin(j\theta), \tag{7}$$

where a number r of harmonics has been taken into consideration.

By operating in this way a large number of harmonics can be taken into consideration and their dependence on α and β needs not to be approximated by using power series. To obtain a very good approximation, the FFT can be computed by using 4096 or more points without substantial increase of computation time. A comparison between the results obtained using the traditional formulae (e.g., those reported in reference [3]) and the numerical approach is reported in reference [4].

To take into account the phase δ_c between the generic crank and a fixed angular reference, e.g., the phase of the first crank, the expression for the equivalent moment of inertia can be transformed as

$$J_{eq} = J_{c_0} + \sum_{j=1}^{r} J^*_{c_j}\cos(j\theta) + \sum_{j=1}^{r} J^*_{s_j}\sin(j\theta), \tag{8}$$

where

$$J^*_{c_j} = J_{c_j}\cos(j\delta_c) + J_{s_j}\sin(j\delta_c), \qquad J^*_{s_j} = J_{s_j}\cos(j\delta_c) - J_{c_j}\sin(j\delta_c). \tag{9}$$

If the cylinders are not all in the same plane, as occurs in V or radial engines, δ_c must be replaced by $\delta_c - \delta^*$, where δ^* is the angle between the plane containing the generic cylinder and a reference plane, e.g., that containing the first cylinder.

By introducing the imaginary unit $\mathrm{i} = \sqrt{-1}$, equation (8) can be transformed as

$$J_{eq} = J_{c_0} + \sum_{j=1}^{r} A_j \, \mathrm{e}^{\mathrm{i}j\theta} + \sum_{j=1}^{r} B_j \, \mathrm{e}^{-\mathrm{i}j\theta}, \tag{10}$$

where

$$A_j = (J^*_{c_j} - \mathrm{i}J^*_{s_j})/2, \qquad B_j = (J^*_{c_j} + \mathrm{i}J^*_{s_j})/2. \tag{11}$$

If the angular velocity ω of the crankshaft is kept constant, the crank angle θ can be expressed as

$$\theta(t) = \omega t + \phi(t), \tag{12}$$

where ϕ is an angular displacement linked with the torsional deformation of the shaft. By assuming the latter to be the generalized co-ordinate of the crank for torsional motions, the equation of motion obtained through the Lagrange equation is

$$J_{eq}\ddot{\phi} + \tfrac{1}{2}(\omega + \dot{\phi})^2 \, (\mathrm{d}J_{eq}(\theta)/\mathrm{d}\theta) = M, \tag{13}$$

where the torque M is the total torque acting on the crank and includes the elastic and damping reaction of the shaft and the forces acting on the piston.

Equation (13) is usually drastically simplified by assuming a constant value for the equivalent moment of inertia in the first term $J_{eq}\ddot{\phi}$, for instance the constant term J_{c_0} of series (7). $\dot{\phi}$ is then neglected in the sum $(\omega + \dot{\phi})$ in the second term and ϕ is neglected in the expression for θ in the derivative $\mathrm{d}J_{eq}/\mathrm{d}\theta$. The simplified differential equation with constant coefficients so obtained is

$$\bar{J}_{eq}\ddot{\phi} = M - \tfrac{1}{2}\omega^2[\mathrm{d}J_{eq}(\omega t)/\mathrm{d}(\omega t)]. \tag{14}$$

The known function of angle (ωt) and hence of time appearing in the second term can be considered as a forcing function applied to the system and can be computed either by using the numerical values of the coefficients of the series (7) or by using the formulae for the coefficient of the series for functions $f_1(\theta)$ and $f_2(\theta)$ reported in the literature:

$$\frac{1}{2}\,\omega^2\left[\frac{\mathrm{d}J_{eq}(\omega t)}{\mathrm{d}(\omega t)}\right] = \frac{1}{2}\,\omega^2 \sum_{j=1}^{r} j\,[-J_{c_j}\sin(j\omega t) + J_{s_j}\cos(j\omega t)]. \tag{15}$$

The use of equation (14) leads to approximated results, which can be affected by large errors, particularly when the inertia of the reciprocating masses is large but is up to now the only viable approach when a complex system, such as a multi-cylinder engine, has to be dealt with. Actually, there is little difficulty to assemble the equations related to the various cranks and the other parts of the equivalent system, and to introduce also parts connected to the crankshaft via gear trains. One of the codes which can be used for the study of the free and forced vibrations of a system built in this way is DYNROT, finite element code for rotordynamics developed by the authors [5].

2.2. BEYOND THE TRADITIONAL APPROACH

To obtain the differential equation with constant coefficients (14) quite crude assumptions have been made. To circumvent this problem the equivalent moment of inertia, which is a function of $(\omega t + \phi)$, can be expressed by the power series

$$J_{eq}(\omega t + \phi) = J_{eq}(\omega t) + \phi \frac{\partial J_{eq}(\omega t)}{\partial(\omega t)} + \frac{\phi^2}{2} \frac{\partial^2 J_{eq}(\omega t)}{\partial(\omega t)^2} + \frac{\phi^3}{3!} \frac{\partial^3 J_{eq}(\omega t)}{\partial(\omega t)^3} + \cdots . \tag{16}$$

As a consequence, the equation of motion can be expressed in the form

$$\ddot{\phi}\left[J_{eq}(\omega t) + \phi \frac{\partial J_{eq}(\omega t)}{\partial(\omega t)} + \phi^2 \frac{1}{2} \frac{\partial^2 J_{eq}(\omega t)}{\partial(\omega t)^2} + \phi^3 \frac{1}{3!} \frac{\partial^3 J_{eq}(\omega t)}{\partial(\omega t)^3} + \ldots \right]$$

$$+ \frac{1}{2}(\omega + \dot{\phi})^2 \left[\frac{\partial J_{eq}(\omega t)}{\partial(\omega t)} + \phi \frac{\partial^2 J_{eq}(\omega t)}{\partial(\omega t)^2} + \phi^2 \frac{1}{2} \frac{\partial^3 J_{eq}(\omega t)}{\partial(\omega t)^3} + \phi^3 \frac{1}{3!} \frac{\partial^4 J_{eq}(\omega t)}{\partial(\omega t)^4} + \ldots \right] = M. \tag{17}$$

Equation (17) is a non-linear differential equation with periodic coefficients in ϕ. It tends to the exact equation of motion of the crank system if the number of terms considered tends to infinity. For the study of the small oscillations of the system it can be linearized, obtaining

$$J_{eq}(\omega t)\ddot{\phi} + \dot{\phi}\omega \frac{\partial J_{eq}(\omega t)}{\partial(\omega t)} + \phi \frac{\omega^2}{2} \frac{\partial^2 J_{eq}(\omega t)}{\partial(\omega t)^2} = -\frac{\omega^2}{2} \frac{\partial J_{eq}(\omega t)}{\partial(\omega t)} + M. \tag{18}$$

Such linearization is performed by neglecting terms in ϕ^2, $\dot{\phi}^2$, $\dot{\phi}\phi$ and $\ddot{\phi}\phi$ and allows one to study the small motion about the condition $\phi = 0$. Note that the range in which this linearization holds becomes narrower if high frequency motions are considered, but nevertheless always exists.

By introducing expression (10) for the equivalent moment of inertia into equation (18) it follows that

$$J_{c_0}\ddot{\phi} + \ddot{\phi} \sum_{j=1}^{r} [A_j \, e^{ij\omega t} + B_j \, e^{-ij\omega t}] + i\dot{\phi}\omega \sum_{j=1}^{r} j \, [A_j \, e^{ij\omega t} - B_j \, e^{-ij\omega t}]$$

$$-\frac{1}{2}\phi\omega^2 \sum_{j=1}^{r} j^2 [A_j \, e^{ij\omega t} + B_j \, e^{-ij\omega t}] = -\frac{1}{2} i\omega^2 \sum_{j=1}^{r} j \, [A_j \, e^{ij\omega t} - B_j \, e^{-ij\omega t}] + M. \tag{19}$$

Equation (19) is the linearized mathematical model of the crank and can be assembled, together with the other elements (beams, masses, dampers, etc.) to build the model of the whole system by using the conventional techniques used in finite element procedures. The resulting model of the whole system consists of a number of linear second order differential equations with coefficients which are periodic in time with period $2\pi/\omega$. The forcing terms are themselves periodic, with the same period of the coefficients (gas compressors or two-stroke cycle engines) or period $4\pi/\omega$ (four-stroke cycle engines).

The result of this assembly procedure is the following set of equations

$$[J]\{\ddot{\phi}\} + \sum_{j=1}^{r} \big[[A_j] \, e^{ij\omega t} + [B_j] \, e^{-ij\omega t}\big]\{\ddot{\phi}\} + i\omega \sum_{j=1}^{r} j \, \big[[A_j] \, e^{ij\omega t} - [B_j] \, e^{-ij\omega t}\big]\{\dot{\phi}\}$$

$$+ [C]\{\dot{\phi}\} - \frac{1}{2}\omega^2 \sum_{j=1}^{r} j^2 \big[[A_j] \, e^{ij\omega t} + [B_j] \, e^{-ij\omega t}\big]\{\phi\} + [K]\{\phi\}$$

$$= -\frac{1}{2}\,\mathrm{i}\omega^2 \sum_{j=1}^{r} j[\{A_j\}\ \mathrm{e}^{\mathrm{i}j\omega t} - \{B_j\}\ \ \mathrm{e}^{-\mathrm{i}j\omega t}] + \{M_m\}, \tag{20}$$

where $[J]$, $[C]$, $[K]$ and $\{M_m\}$ are the usual inertia, damping and stiffness matrices and driving torque vector obtained from the usual model with coefficients which are constant in time. The first is always diagonal, the second is often of the same type and in most cases the third is tridiagonal. Matrices $[A_j]$ and $[B_j]$ are diagonal matrices containing coefficients A_j and B_j, the same coefficient are listed in vectors $\{A_j\}$ and $\{B_j\}$.

From the homogeneous equation obtained by neglecting all forcing terms at the right side of equation (19) the stability of the system can be studied. By considering also the forcing terms it is possible to study the forced response. In all cases an approximated solution expressed as a truncated series of trigonometric or complex exponential terms can be assumed. The procedure is not dissimilar, although more complex, from that described in reference [3] and [6] for the study of the flexural dynamics of anisotropic rotors.

2.3. STABILITY ANALYSIS OF A SIMPLIFIED SINGLE-CYLINDER MACHINE

In the literature it is possible to find the stability analysis of a simplified single-cylinder machine made of a crank, with its connecting rod and piston, connected to a large flywheel through a torsionally compliant shaft [7]. Let K be the stiffness of the shaft and C the damping coefficient of a viscous damper acting on the crank and supplying a torque proportional to its angular velocity. If the moment of inertia of the flywheel is large enough, the system can be studied as a single degree of freedom system, constrained at the flywheel location. The torque M which has to be introduced into the equation of motion is

$$M = -K\phi - C(\omega + \dot{\phi}) + M_m(\omega t), \tag{21}$$

where ϕ is the torsion of the shaft and M_m is the driving torque due to the pressure on the piston. In reference [7] the length of the connecting rod is assumed to be very large and then $\alpha = \beta = 0$. The expression of the moment of inertia is simply

$$J_{eq} = \bar{J}_{eq}\{1 - \varepsilon \cos[2(\omega t + \phi)]\}, \tag{22}$$

where

$$\bar{J}_{eq} = J_d + (m_1 + (m_2 + m_p)/2)r^2, \qquad \varepsilon = r^2(1/\bar{J}_{eq})(m_2 + m_p)/2. \tag{23}$$

Upon introducing the natural frequency of the average equivalent system $\lambda_0 = \sqrt{K/\bar{J}_{eq}}$ and the damping ratio $\zeta = C/(2\sqrt{K\bar{J}_{\mathrm{eq}}})$, the equation of motion (19) reduces to

$$\ddot{\phi}[1 - \varepsilon\cos(2\omega t)] + 2\varepsilon\omega^2\phi\cos(2\omega t) + 2\varepsilon\omega\dot{\phi}\sin(2\omega t)$$
$$+ 2\zeta\lambda_0(\omega + \dot{\phi}) + \lambda_0^2\phi = -\varepsilon\omega^2\sin(2\omega t) + \lambda_0^2 M_m(t)/K. \tag{24}$$

Equation (24) coincides with the equation obtained in reference [7]. In the mentioned work the stability was studied by integrating numerically the homogeneous equation (24) with different values of parameter ε, damping ratio ζ and ratio λ_0/ω. The same system has been studied in reference [4] by using an approach similar to that described below for the general multi-cylinder machine obtaining the same results.

2.4. STABILITY ANALYSIS

Consider the homogeneous equation obtained by neglecting the right side of equation (20). Its general solution can be approximated as

$$\{\phi\} = \sum_{k=-m}^{m} \{q\}_k \, \mathrm{e}^{\mathrm{i}(\lambda + k\omega)t}. \tag{25}$$

The solution expressed by equation (25) tends to the exact solution when m tends to infinity. By substituting equation (25) into the homogeneous equation of motion obtained from equation (20), the following algebraic equation is readily obtained:

$$
\begin{aligned}
&-\sum_{k=-m}^{m}\left\{(\lambda + k\omega)^2\left[[J] + \sum_{j=1}^{r}([A_j]\,\mathrm{e}^{\mathrm{i}j\omega t} + [B_j]\,\mathrm{e}^{-\mathrm{i}j\omega t})\right]\{q\}_k\,\mathrm{e}^{\mathrm{i}(\lambda + k\omega)t}\right\} \\
&+\sum_{k=-m}^{m}\left\{(\lambda + k\omega)\left[\mathrm{i}[C] - \omega\sum_{j=1}^{r} j([A_j]\,\mathrm{e}^{\mathrm{i}j\omega t} - [B_j]\,\mathrm{e}^{-\mathrm{i}j\omega t})\right]\{q\}_k\,\mathrm{e}^{\mathrm{i}(\lambda + k\omega)t}\right\} \\
&+\sum_{k=-m}^{m}\left\{\left[[K] - \frac{1}{2}\omega^2\sum_{j=1}^{r} j^2([A_j]\,\mathrm{e}^{\mathrm{i}j\omega t} + [B_j]\,\mathrm{e}^{-\mathrm{i}j\omega t})\right]\{q\}_k\,\mathrm{e}^{\mathrm{i}(\lambda + k\omega)t}\right\} = \{0\}.
\end{aligned} \tag{26}
$$

Equation (26) can be considered an eigenproblem in λ. Its solution can be performed by harmonic balancing; the relevant computations being quite intricate but conceptually straightforward. After harmonic balancing the following formulation of the eigenproblem is obtained:

$$[-\lambda^2[J^*] + \mathrm{i}\,\lambda[C^*] + [K^*]]\{q^*\} = \{0\}. \tag{27}$$

The size of all matrices and vectors in equation (27) is $(2m+1)n$, where m is the number of harmonics which are balanced and n is the number of degrees of freedom of the system. The augmented inertia matrix $[J^*]$ is

$$[J^*] = \begin{bmatrix}
— & — & — & — & — & — & — \\
— & [J] & [B_1] & [B_2] & [B_3] & [B_4] & — \\
— & [A_1] & [J] & [B_1] & [B_2] & [B_3] & — \\
— & [A_2] & [A_1] & [J] & [B_1] & [B_2] & — \\
— & [A_3] & [A_2] & [A_1] & [J] & [B_1] & — \\
— & [A_4] & [A_3] & [A_2] & [A_1] & [J] & — \\
— & — & — & — & — & — & —
\end{bmatrix}. \tag{28}$$

In a similar way, the other matrices can be partitioned in submatrices of order n. The generic submatrix in position pk is

$$[C^*_{kk}] = \mathrm{i}\,2k\omega[J] + [C]$$

$$[K^*_{kk}] = -k^2\omega^2[J] + \mathrm{i}\,k\omega[C] + [K], \tag{29}$$

for the submatrices on the main diagonal,

$$[C^*_{pk}] = \mathrm{i}(p+k)\omega[B_{k-p}], \qquad [K^*_{pk}] = -\tfrac{1}{2}(p^2 + k^2)\omega^2\,[B_{k-p}] \tag{30}$$

if $k > p$ and

$$[C^*_{pk}] = \mathrm{i}(p+k)\omega[A_{p-k}], \qquad [K^*_{pk}] = -\tfrac{1}{2}(p^2 + k^2)\,\omega^2\,[A_{p-k}] \tag{31}$$

if $k < p$. Note that both p and k take values between $-m$ and m: the central submatrices have then indices 0,0.

By resorting to a state–space approach, the equation yielding the eigenproblem in standard form is

$$\det\left[\begin{bmatrix} [J^*]^{-1}[C^*] & [J^*]^{-1}[K^*] \\ [I] & [0] \end{bmatrix} - \lambda\begin{bmatrix} [I] & [0] \\ [0] & [I] \end{bmatrix}\right] = 0. \tag{32}$$

From the definition of λ it follows that the frequency of the free motion of the system is its real part while the imaginary part of λ is the decay rate, which must be positive to insure stability. The fields of instability are then characterized by negative values of the imaginary part of λ.

2.5. FORCED RESPONSE

The driving torque acting on the various cranks is usually expressed in the form

$$\{M_m\} = \sum_{j=1}^{s}\{M^*_{m_{cj}}\}\cos(j\omega' t) + \sum_{j=1}^{s}\{M^*_{m_{sj}}\}\sin(j\omega' t), \tag{33}$$

where

$$\begin{aligned} \{M^*_{m_{cj}}\} &= \{M_{m_{cj}}\cos(j\delta_h) + M_{m_{sj}}\sin(j\delta_h)\}, \\ \{M^*_{m_{sj}}\} &= \{M_{m_{sj}}\cos(j\delta_h) - M_{m_{cj}}\sin(j\delta_h)\} \end{aligned} \tag{34}$$

and ω' is equal to ω in the case of gas compressors or two-stroke cycle engines or $\omega/2$ for four-stroke cycle engines. In this case angles δ_h are the phase angles of the first harmonic of the driving torque, as available from the phase angle diagram.

In case $\omega = \omega'$ and the same number of harmonics are taken for expressing the moment of inertia and the driving torque ($r = s$), the two terms of the right side of equation (20) contain the same harmonic terms. If on the contrary $\omega = 2\omega'$, as occurs in the case of four-stroke cycles engines, some modifications are needed. As a first point the series for the moments of inertia of the cranks can be rewritten by using ω' as the fundamental frequency simply by setting new values of A_j and B_j equal to zero for odd values of j and equal to the previously defined $A_{j/2}$ and $B_{j/2}$ for even values of j. If $r < s/2$ a number of coefficients equal to zero are then inserted.

The forced response of the system can be expressed in the form

$$\{\phi\} = \sum_{k=-m}^{m}\{q\}_k\,\mathrm{e}^{\mathrm{i}k\omega' t}. \tag{35}$$

Note that in this case the argument of the exponential is imaginary instead of being complex and the response can be expressed directly by using sines and cosines. Even if this implies rewriting equation (20) in a different form, it has the advantage of yielding results in the form of the amplitudes of the 'in phase' and 'in quadrature' components of the various harmonics in a way immediately comparable with the results of the traditional approach. Equation (20) can then be rewritten as

$$\left([J] + \sum_{j=1}^{r}[[J_{cj}]\cos(j\omega' t) + [J_{sj}]\sin(j\omega' t)]\right)\{\ddot{\phi}\}$$

$$+\left([C]-\omega'\sum_{j=1}^{r} j[[J_{c_j}]\sin(j\omega't)-[J_{s_j}]\cos(j\omega't)]\right)\{\dot{\phi}\}$$

$$+\left([K]-\frac{1}{2}\omega'^2\sum_{j=1}^{r} j^2[[J_{c_j}]\cos(j\omega't)+[J_{s_j}]\sin(j\omega't)]\right)\{\phi\}$$

$$=\omega'^2\sum_{j=1}^{r} j[[J_{c_j}]\sin(j\omega't)-[J_{s_j}]\cos(j\omega't)]+\{M_m\}. \tag{36}$$

Note that equation (36) is general, as in the case of two stroke cycle engines all coefficients of odd harmonics vanish and then the fundamental frequency is $2\omega'$.

The solution of equation (36) can be approximated as

$$\{\phi\}=\sum_{k=1}^{m}\{\phi_{c_k}\}\cos(k\omega't)+\sum_{k=1}^{m}\{\phi_{s_k}\}\sin(k\omega't). \tag{37}$$

Again the solution tends to the exact one when m tends to infinity. By substituting equation (37) into the equation of motion (36), an algebraic equation is readily obtained. By balancing the various harmonics in $\cos(k\omega't)$ and $\sin(k\omega't)$, it is possible to obtain the equation

$$[K^*_{dyn}]\{\phi^*\}=\{M^*\}. \tag{38}$$

If it is assumed that $r=s=m$, i.e., that the same number of harmonics is used to express the moments of inertia (with fundamental frequency ω'), the driving torque and the response, the matrices and vectors in equation (38) have a size equal to $2mn$. The set of equations (38) can be partitioned into m subsets as

$$\begin{bmatrix}[K^*_{11}] & [K^*_{12}] & [K^*_{13}] & \cdots & [K^*_{1m}]\\ [K^*_{21}] & [K^*_{22}] & [K^*_{23}] & \cdots & [K^*_{2m}]\\ [K^*_{31}] & [K^*_{32}] & [K^*_{33}] & \cdots & [K^*_{3m}]\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ [K^*_{m1}] & [K^*_{m2}] & [K^*_{m3}] & \cdots & [K^*_{mm}]\end{bmatrix}\begin{Bmatrix}\{\phi^*_1\}\\ \{\phi^*_2\}\\ \{\phi^*_3\}\\ \vdots\\ \{\phi^*_m\}\end{Bmatrix}=\begin{Bmatrix}\{M^*_1\}\\ \{M^*_2\}\\ \{M^*_3]\\ \vdots\\ \{M^*_m\}\end{Bmatrix}, \tag{39}$$

where

$$[K^*_{kk}]=\begin{bmatrix}-\omega'^2k^2[J]+[K] & \omega'k[C]\\ -\omega'k[C] & -\omega'^2k^2[J]+[K]\end{bmatrix} \tag{40}$$

and

$$[K^*_{pk}]=\begin{bmatrix}u[J_c]_{|k-p|} & \pm u[J_s]_{|k-p|}\\ \mp u[J_s]_{|k-p|} & u[J_c]_{|k-p|}\end{bmatrix}. \tag{41}$$

The upper sign is used if $k>p$ and the lower one if $k<p$ and

$$u=(\omega'/2)[k^2-k(k-p)+(k-p)^2]. \tag{42}$$

The vectors are

$$\{\phi^*\}_k = \begin{Bmatrix} \{\phi_c\}_k \\ \{\phi_s\}_k \end{Bmatrix}, \qquad \{M^*\}_k = \begin{Bmatrix} -\omega'^2 k\{J_s\}_k + \{M_c\}_k \\ \omega'^2 k\{J_c\}_k + \{M_s\}_k \end{Bmatrix}. \tag{43}$$

If the equivalent moment of inertia is assumed to be constant, the set of equations (38) uncouples into m independent sets of $2n$ equations, each one supplying the response to a particular harmonic. They are identical to those obtained through the conventional approach [3].

Some simplifications are however always possible. In the case of four-stroke cycle engines the sets of equations related to the odd harmonics can be uncoupled and solved separately. The set (38) can thus be written to include only even harmonics (i.e., p and k even).

In the case of two-stroke cycle engines and compressors, only even harmonics exist and the set (38) can be greatly simplified. Note that also in this case equation (38) and those following hold only if $\omega' = \omega/2$; if other choices are made the coefficients must be modified.

3. EXAMPLES

3.1. SIMPLIFIED SINGLE-CYLINDER MACHINE

Consider a simplified single-cylinder machine of the same type described in section 2.3., but release the assumption that the connecting rod is very long (i.e., that α is close to zero). Without lack of generality, assume that the moment of inertia J_0 is negligible: i.e., that the mass distribution of the connecting rod is such as to be modelled as two masses located at the crank and wrist pins. Under this assumption, the dynamic behaviour of the system is function of a small number of parameters, namely ε, $\zeta = C/(2\sqrt{K\bar{J}_{eq}})$, $\omega/\lambda_0 = \omega/\sqrt{K/\bar{J}_{eq}}$, α and β. Note that now the definition of $\bar{J}_{eq}$ given in equation (23) does not hold, while the definition of ε is still valid.

Even if detailed study of the stability is not possible in general, owing to the increased number of parameters, some conclusions can be drawn. The decay rate $\mathfrak{I}(\lambda)$ is plotted as a function of the nondimensional speed in Figure 2 for $\varepsilon = 0{\cdot}4$ and $\beta = 0$ and various values of the damping ratio ζ. Two values of α, namely 0 and 0·2, have been considered.

If $\alpha = 0$ the model is identical to that studied in references [4, 7] and, if no damping is considered, it shows the two fields of instability for ω/λ_0 equal to about 1 and 0·5 already identified in reference [7]. As is well known, the first of the two is far stronger, in the sense that the instability range is larger, the negative value of the decay rate has a greater absolute value and a greater damping is required to overcome the instability.

If α is not equal to zero, the picture changes as a third field of instability is present at ω/λ_0 equal to about 2/3. This new instability is weaker than that occurring at $\omega/\lambda_0 = 1$, but far stronger than that occurring at $\omega/\lambda_0 = 0{\cdot}5$. Although not visible in the figure, an analysis of the numerical results shows that other very weak instability ranges can occur at lower values of ω/λ_0 (1/3, 1/4, etc.), but a very small damping is able to make them disappear.

The instability ranges are shown in Figure 3 for systems with different values of ε. The curves are characterized by different values of α. The largest field, starting at $\omega/\lambda_0 = 1$, is not much affected by the value of α (the curves are for $\alpha = 0$, 0·25, 0·50). The field starting at $\omega/\lambda_0 = 2/3$ exists only if $\alpha \neq 0$ and is very sensitive to the value of the parameter (the curves are for $\alpha = 0{\cdot}1$, 0·2, 0·3, 0·4 and 0·5). While the other fields move slightly to the left with increasing α, the present one widens. The last field, starting at $\omega/\lambda_0 = 1/2$, is very narrow and again has a weak dependence on the parameter α.

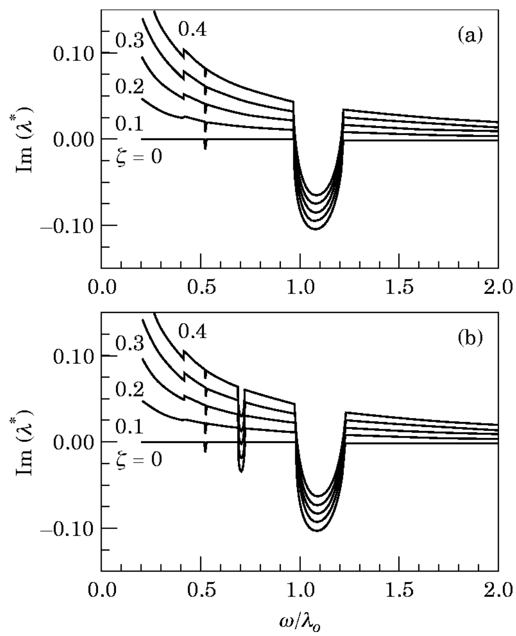

Figure 2. Minimum value of the decay rate as function of the non-dimensional speed for systems with different damping ($\varepsilon = 0{\cdot}4$). Simplified model with $\alpha = 0$ (a) and model with $\alpha = 0{\cdot}2$ (b) In the second case a new field of instability is present. All values of damping are lower than that needed to make the larger field of instability disappear. All curves have been obtained with $\beta = 0$.

The results here obtained, although without considering the effects of J_0 and β, are nevertheless quite general: the important fact is the presence of harmonics other than the second one in the series for J_{eq} and not the exact value of the parameters. It must however be remembered that the presence of β would introduce sine terms, and not just cosine terms, into the series.

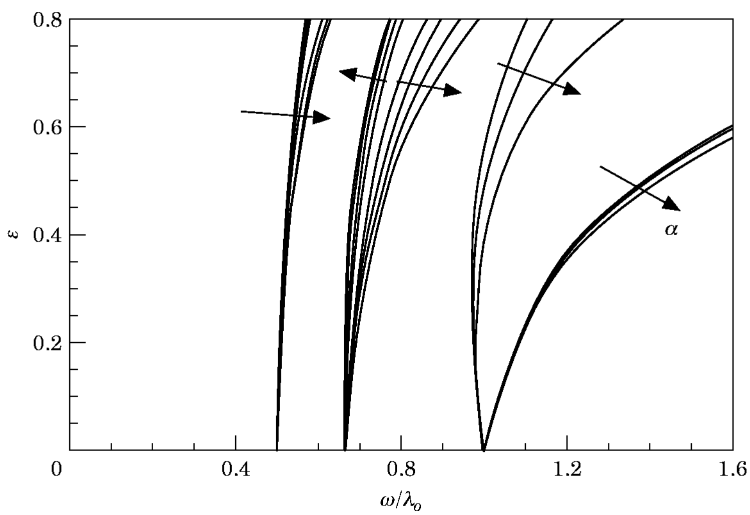

Figure 3. Boundaries of the fields of instability for the three fields identified in Figure 2 with $\zeta = 0$. The values of α considered are: 0, 0·1, 0·2, 0·3, 0·4 and 0·5 for the field on the left, 0·1, 0·2, 0·3, 0·4 and 0·5 for the central field and 0, 0·25 and 0·5 for the field on the right.

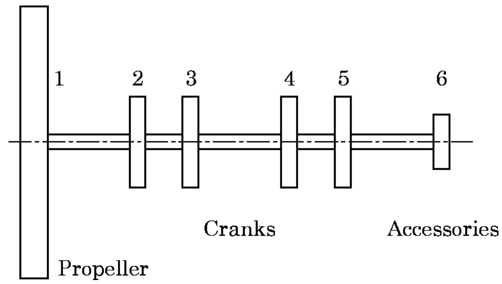

Figure 4. Equivalent system of a four-cylinder engine driving an aerial propeller. Inertia at node 6 models the accessories driven through a gearbox.

3.2. DYNAMIC ANALYSIS OF MULTI-CYLINDER ENGINE

Consider a flat four-cylinder, four stroke cycle engine Lycoming O-360-A3A coupled to a Sensenich 76EM8S5-0-58 fixed pitch propeller as used on the four-seater light aircraft Robin DR400/180 R. All geometrical and inertial data of the machine relevant for the analysis of the dynamic torsional behaviour of the crankshaft have been either computed or measured in reference [8]. In particular, the stiffness of the various sections of the shaft were measured on a shaft which was discarded from service. The rotating and reciprocating parts were modelled as an equivalent system with six degrees of freedom (see Figure 4). The inertia at node 1 simulates the propeller while that at node 6 simulates the accessories and their gear train.

The coefficients of the harmonics of driving torque were evaluated as functions of the mean indicated pressure p_{mi} by using the formula [2]

$$M_{m_k} = p_{mi} r A 25/\left[50\sqrt{(2/k) + 5k^2}\right]. \tag{44}$$

The disadvantage of this approach is that the formula supplies the coefficients of the harmonics but not their phases. However, in reference [2] the phases of the harmonics with frequencies 2, 4, 6 and 8 ω' are stated and this allows one to add the torques due to the inertia of the reciprocating parts to those due to the pressure of the working fluid. In reference [8] several computations were run with different laws linking the mean indicated pressure with the speed. Here just one of the cases studied will be reported. The mean indicated pressure is then expressed as

$$p_{mi} = -3{\cdot}2283 + 4{\cdot}1081 \times 10^{-2}\omega - 1{\cdot}5402 \times 10^{-4}\omega^2 + 2{\cdot}2366 \times 10^{-7}\omega^3.$$

In a similar way, the damping of the system was evaluated by adding a viscous damper in each crank, with a coefficient equal to

$$C_{eq} = k' A r^2, \tag{45}$$

where the coefficient k' was assumed as 27150 Nsm^{-3}. The formula and the numerical values were taken from reference [2]. In a similar way also the damping of the propeller was considered.

All computations were performed by using the DYNROT 6·0 code, developed by the authors, in which the formulation to take into account the variation in time of the equivalent moment of inertia was added.

The first step in the analysis is the computation of the natural frequencies of the constant-inertia equivalent system. The following values were obtained for the first two non-zero natural frequencies: $\lambda_{n1} = 1900$ rad/s $= 302$ Hz, $\lambda_{n_2} = 5705$ rad/s $= 908$ Hz.

The forced torsional response, in terms of dynamic shear stresses in the propeller shaft and in the journal of the central bearing, computed using the conventional approach, is shown in Figure 5 (dashed lines). A sharp peak, corresponding to the resonance of the 16th harmonic of the driving torque, occurs in the working range, at 237 rad/s (2270 rpm). This speed is roughly in the centre of the range 2150–2350 rpm at which continuous running is prohibited by the manufacturer.

For the study of the stability of the system, a plot of the natural frequencies and of the decay rates versus the speed must be plotted. Alternatively it is possible to plot a root locus. The first is shown in Figure 6 while the second is in Figure 7. Note that while in the first one the complex frequency λ is reported, in the second one the Laplace variable $s = i\lambda$ has been plotted, as customary. Both have been plotted by taking into consideration four harmonics of the equivalent moment of inertia: as the number of master degrees of freedom is six; all relevant matrices have 54 rows and columns and a set of eigenproblems of order 108 (in the state space) has been solved.

In Figure 6 a number of branches starting from the origin is visible. They are linked with rigid-body modes and actually the corresponding eigenvectors are constant. More precisely, as each eigenvector is made of parts corresponding to the various harmonics, they are formed by parts which are constant, harmonic by harmonic. At low speed each mode has a positive decay rate (or negative real part of s) and hence they are stable. At increasing speed some of them show an unstable nature, with negative decay rate, even if the absolute value of the decay rate is very small (of the order of $0{\cdot}2\ s^{-1}$). This instability has nothing to do with a possible torsional instability of the system, as the mode is not a torsional vibration. It is the opinion of the authors that it is simply linked with the presence of a residual periodic irregularity, i.e., non-constant angular velocity, which is present in spite of the large moment of inertia of the propeller. The absolute value is so low that it appears to be simply linked with numerical approximations.

All other modes have positive decay rates, as could be expected because the engine works in a speed range which is far lower than the ranges in which instability can be expected [7]. When the speed tends to zero, the frequency of the torsional vibration does not tend to the above mentioned values obtained from the simplified model. Actually, the moment of inertia of the crank-piston systems depends on the position of the shaft, varying for

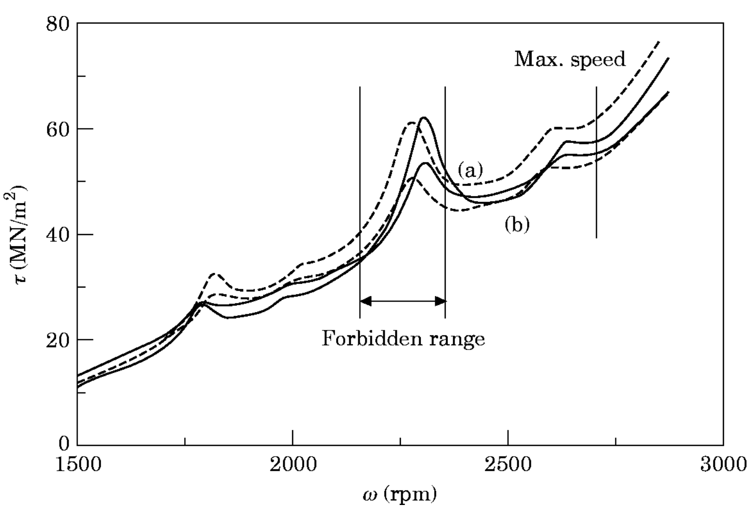

Figure 5. Forced torsional response in terms of dynamic stressing of the crankshaft. Results for the propeller shaft (a) and the central journal (b), computed using both the conventional (– – –) and present (——) approaches.

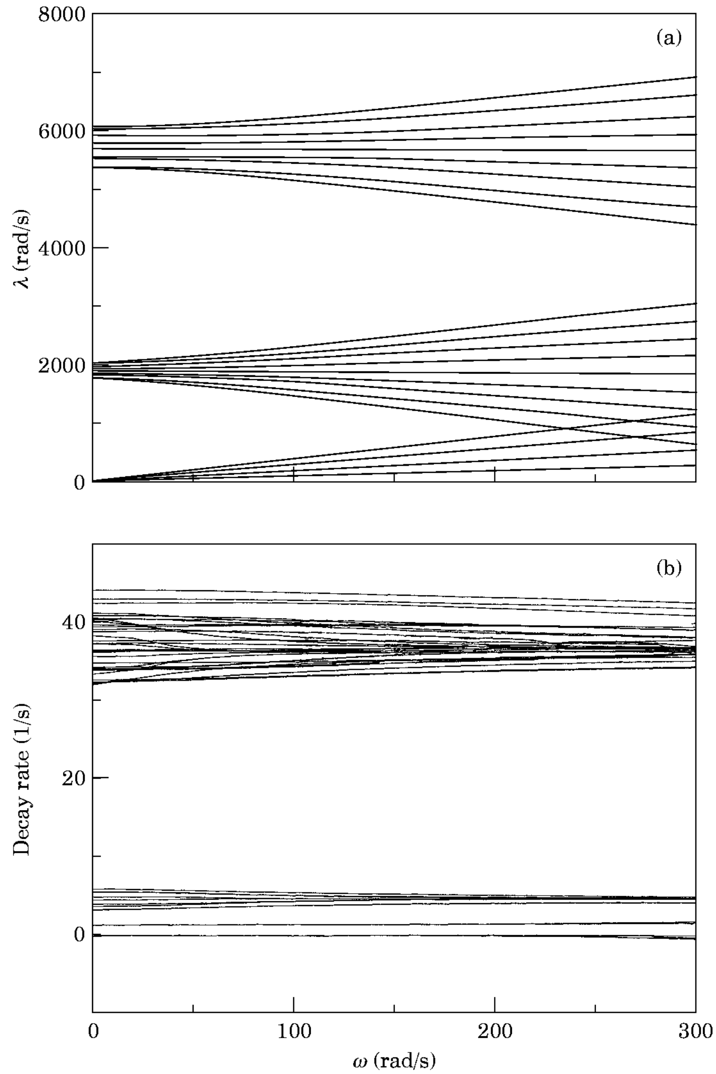

Figure 6. Plot of the natural frequencies (a) and of the decay rates (b) as functions of the speed.

example between 0·01253 kgm^2 and 0·017375 kgm^2 (with a value of $\bar{J}_{eq} = 0{\cdot}014637$ kgm^2) for the first crank. Correspondingly the natural torsional frequency at standstill varies between 1751 and 2044 rad/s for the first mode (1899 rad/s computed with the simplified model). All the values obtained lay within these ranges, even if to get exactly all the span, an infinity of harmonics needs to be computed. As an example the ranges obtained with five harmonics for the first and second mode are respectively 1797–2026 rad/s and 5384–6101 rad/s. With 10 harmonics (involving the solution of an eigenproblem or order 252) the ranges are respectively 1790–2037 rad/s and 5358–6129 rad/s. Understandably, the convergence to the full ranges is slow.

The decay rates of all harmonics for the second and third modes are of about 35 s^{-1}, with small variations with the engine speed, showing a good stability of the system.

Finally, the forced response was computed again, with account taken of the variability of the equivalent moment of inertia. The results, in terms of total dynamic shear stresses in the propeller shaft and the journal of the central bearing, obtained by considering 20 harmonics of the moment of inertia, are reported in Figure 5 (full lines). The results do not differ much from those obtained by using the conventional procedure, but this cannot be generalized, as the engine studied is relatively slow and has a very stiff shaft (also owing to the general configuration) and light reciprocating parts, leading to high values of the torsional natural frequencies. An interesting feature is however a shift of the resonance peaks.

4. CONCLUSIONS

The formulation presented here for the study of the free behaviour of the system allows one to compute the natural frequencies with a greater precision, at the cost of an increase of the size of the eigenproblem to be solved. Instead of dealing with matrices with a number of rows and columns smaller or slightly higher than 10, an eigenproblem of an order which can be easily of several hundreds has to be solved at each value of the speed. However with the use of computers, even a PC machine, this can be a minor problem: all computations needed for the examples shown were performed with use of never more than several minutes of computer time on a 486 class machine.

An example dealing with a simplified single cylinder machine showed that the instability ranges obtained in the literature [7], obtained by using simpler models, were obtained with minor modifications, but a new field of instability, up to now unmentioned, was found. This instability occurred at a speed corresponding to about 2/3 of the natural frequency of the system: i.e., on the intersection of the line $\lambda = 3\omega/2$ on the Campbell diagram. While weaker than that found at a speed equal to the natural frequency (on the line $\lambda = \omega$), it is far stronger than that occurring on the line $\lambda = 2\omega$.

The analysis of the stressing of the shaft due to the forcing functions acting on the pistons is simpler. In this case the difference with the conventional approach is less marked in terms of computation time. The relevant equations are similar, but while in the usual computations the sets of equations yielding the response to the various harmonics of the

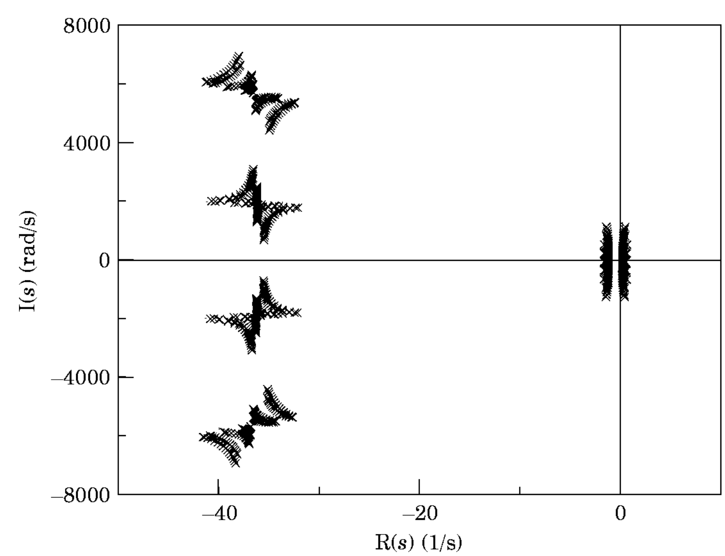

Figure 7. Root locus. Note that the Laplace variables $s = i\lambda$ has been used instead of the complex frequency λ as in Figure 6.

forcing function are uncoupled, the present computation is based on a set of equations all coupled with each other. In the case of the four-cylinder engine shown, at each speed a set of 240 equations has been solved (20 harmonics in sine and cosine on a system with six degrees of freedom), which is a trivial task for any modern computing machine.

The example shows that the difference with the conventional computation is not great, with a certain increase of the stressing and a shift of the frequencies at which the peaks occur. This result however cannot be generalized, and there is the possibility that for a fast machine or a machine with greater reciprocating masses the two approaches give more different results.

The equations obtained are now implemented in a computer code which allows performance of the flexural, axial and torsional dynamic analysis of rotors of many different types.

REFERENCES

1. E. J. NESTORIDES 1958 *A Handbook of Torsional Vibration*. Cambridge University Press.
2. K. E. WILSON 1963 *Torsional Vibration Problems*. London: Chapman & Hall.
3. G. GENTA 1994 *Vibration of structures and machines*. New York: Springer. Second edition.
4. D. BASSANI, E. BRUSA, C. DELPRETE, G. GENTA and A. TONOLI 1994 *XXIII Convegno Nazionale AIAS*, *Rende* (*CS*), *Italia*, *September*. Calcolo delle sollecitazioni dinamiche negli alberi a gomiti: influenza delle ipotesi semplificative sui risultati.
5. G. GENTA, C. DELPRETE and D. BASSANI 1996 *Engineering Computations* **13**, 86–109. DYNROT: A finite element code for rotordynamics analysis based on complex co-ordinates.
6. G. GENTA 1988 *Journal of Sound and Vibration* **124**, 27–53, Whirling of unsymmetrical rotors, a finite element approach based on complex co-ordinates.
7. M. S. PARISCHA and W. D. CARNEGIE 1976 *Journal of Sound and Vibration* **46**, Effect of the variable inertia on the damped torsional vibrations of diesel engine systems.
8. E. BRUSA, C. DELPRETE, D. FRANCHIN and G. Genta 1995 *XXIV Convegno Nazionale*, *AIAS*, *Parma*, *Italia*, *September* Analisi dinamica torsionale dell'albero a gomiti di un motore aeronautico.

APPENDIX: SYMBOLS

d	eccentricity of the cylinder
i	imaginary unit ($\mathrm{i} = \sqrt{-1}$)
l	length of connecting rod
m	number of harmonics for the response
n	number of degrees of freedom
p_{mi}	mean indicated pressure
r	crank radius, number of harmonics for the equivalent moment of inertia
s	number of harmonics for the driving torque
t	time
A	area of the piston
C	damping coefficient
$[C]$	viscous damping matrix
$[I]$	identity matrix
J	moment of inertia
$[J]$	inertia matrix
K	stiffness
$[K]$	stiffness matrix
$\{M\}$	torque vector
M_m	driving torque
δ_c	phase angle of the crank
α	parameter ($\alpha = r/l$)
β	parameter ($\beta = d/l$)
ε	parameter defined in equation (23)
δ_h	phase angle of the first harmonic
θ	crank angle
λ	frequency, complex frequency
ϕ	angular displacement
ω	rotational speed

Subscripts

eq	equivalent
I	imaginary part
R	real part